Theory of Cooperation in Complex Social Networks

Bijan Ranjbar-Sahraei[1], **Haitham Bou Ammar**[2], **Daan Bloembergen**[1,3], **Karl Tuyls**[3], **Gerhard Weiss**[1]
[1]Maastricht University, The Netherlands
[2]University of Pennsylvania, PA, United States
[3]University of Liverpool, United Kingdom

Abstract

This paper presents a theoretical as well as empirical study on the evolution of cooperation on complex social networks, following the continuous action iterated prisoner's dilemma (CAIPD) model. In particular, convergence to network-wide agreement is proven for both evolutionary networks with fixed interaction dynamics, as well as for coevolutionary networks where these dynamics change over time. Moreover, an extension to the CAIPD model is proposed that allows to model influence on the evolution of cooperation in social networks. As such, this work contributes to a better understanding of behavioral change on social networks, and provides a first step towards their active control.

Introduction

Modelling the evolution of cooperation in social networks has recently attracted much attention, aiming to understand how individuals work together and influence each other, and how society as a whole evolves over time (Nowak and May 1992; Santos and Pacheco 2005; Ohtsuki et al. 2006; Lazer et al. 2009; Hofmann, Chakraborty, and Sycara 2011). Progress made towards understanding how this evolution comes about has been mostly empirical in nature. Though compelling, deeper insights are better gained from an analytical analysis of the problem.

Meanwhile, the control theory community has developed strong theories for dealing with various types of multi-agent systems. Although many of these theories were initially developed for the analysis of artificial networks such as series of chemical reactors, electrical circuits or robotic swarms (Rosenbrock 1963; Jadbabaie, Lin, and Morse 2003; Ren, Beard, and McLain 2005), they can be extended to the analysis of social networks as well. For instance, Olfati-Saber (2005) uses properties of the Laplacian matrix, a typical tool in the control community for analysis of multi-agent systems, to analyze the information flow in small world networks. Additionally, Liu, Slotine, and Barabási (2011) study controllability, focussing on so-called driving nodes that need to be controlled in various complex networks. Summers and Shames (2013) use theory of nonlinear dynamical systems for modeling and influencing behaviors in specific social networks.

Recently a formal model for understanding the evolution of cooperation in arbitrary social networks has been proposed by Ranjbar-Sahraei et al. (2014). The authors introduce the continuous action iterated prisoner's dilemma (CAIPD) as a general framework for modelling the behavior of individuals in complex networks. Moreover, they show that CAIPD is better able to capture the nature of cooperation in arbitrary networks than existing models. Due to its generalization capabilities and broad spectrum of applications, this paper adopts CAIPD as the formal framework on which an in-depth analytical study is conducted.

Aiming at a broader theoretical understanding of the evolution of cooperation, this paper provides a formal analysis of agreement among individuals in complex social networks, following the CAIPD model. A set of theorems are presented and empirically validated, proving convergence to a final agreement in (co)evolutionary social networks. Additionally, the CAIPD model is extended to allow for the influence on final agreements in such type of networks. In particular, multi-rate adaptation is discussed. It is shown both theoretically and empirically that individuals with the slowest adaptation rates ultimately determine the final agreement. Finally, state-reference tracking is discussed as a special case of the proposed control extension, showing how a varying reference signal can be incorporated to guide the network to any level of agreement.

Background

Continuous-Action Iterated Prisoner's Dilemma

CAIPD (Ranjbar-Sahraei et al. 2014) is adopted for describing the evolution of cooperation in arbitrary complex networks. In CAIPD, N individuals are positioned on a vertices $v_i \in \mathcal{V}$ for $i = 1, 2, \ldots, N$ of a weighted graph $\mathbb{G} = (\mathcal{V}, \mathcal{W})$. The symmetrically weighted $N \times N$ adjacency matrix $\mathcal{W} = [w_{ij}]$, with $w_{ij} \in \{0, 1\}$, describes the i^{th} to j^{th} individual connections with all $w_{ii} = 0$. One of the advantages of CAIPD over other models is the continuous nature in which cooperation and defection levels are modelled. To this end, Ranjbar-Sahraei et al. introduce $x_i \in [0, 1]$ to represent the cooperation level of each individual i, where $x_i = 0$ corresponds to pure defection while $x_i = 1$ represents pure cooperation; an individual pays a cost cx_i while the opponent receives a benefit bx_i, with

$b > c$. This way a defector (i.e., $x_i = 0$) pays no cost and distributes no benefits. Accordingly, the fitness of individual i can be calculated as $f_i = -\text{deg}[v_i]cx_i + b\sum_{j=1}^{N} w_{ij}x_j$, where $\text{deg}[v_i]$ denotes the number of neighbors of v_i. Assuming rationality in behaviors, individual i then has an incentive to adopt its neighbor j's strategy with strength $p_{ij} = w_{ij} \cdot \text{sigmoid}(\beta(f_j - f_i))$ with $\beta > 0$. A network with state vector $\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^{\mathsf{T}}$ and topology $\mathbb{G}$ is defined as $\mathbb{G}_{\boldsymbol{x}} = (\mathbb{G}, \boldsymbol{x})$; the network $\mathbb{G}_{\boldsymbol{x}}$ can then be regarded as a dynamical system, where $\boldsymbol{x}$ evolves according to some nonlinear mapping $\dot{\boldsymbol{x}} = [h_1(\boldsymbol{x}), \ldots, h_N(\boldsymbol{x})]^{\mathsf{T}}$, with $h_i(\boldsymbol{x})$ denoting the dynamics of the i^{th} individual in $\mathbb{G}_{\boldsymbol{x}}$. Precisely, $h_i(\boldsymbol{x}) = \frac{1}{\text{deg}[v_i]}\left[\sum_{j=1}^{N} p_{ij}\left(x_j(t) - x_i(t)\right)\right]$. This dynamical system can be re-written in a standard form by introducing the Laplacian of $\mathbb{G}$, $\mathcal{L}(\cdot)$ as

$$\dot{\boldsymbol{x}}(t) = -\mathcal{L}\left[\boldsymbol{x}(t)\right]\boldsymbol{x}(t), \quad \mathcal{L}_{ij} = \begin{cases} -p_{ij}/\text{deg}[v_i], & i \neq j \\ \sum_{j=1}^{N} p_{ij}/\text{deg}[v_i], & i = j \end{cases} \tag{1}$$

Stochastic Indecomposable and Aperiodic Matrices

This work makes use of lemmas introduced elsewhere, which are briefly discussed. Firstly, however, Stochastic Indecomposable and Aperiodic (SIA) matrices, and $\boldsymbol{\lambda}(\cdot)$ functions, are presented:

Definition 1 (SIA Matrices) *A matrix $\boldsymbol{P}$ with all positive elements is stochastic if all its row sums are $+1$. $\boldsymbol{P}$ is called SIA if $\lim_{n\to\infty} \boldsymbol{P}^n = \mathbf{1}\nu^{\mathsf{T}}$, where ν is a column vector.*

Definition 2 ($\lambda(\cdot)$ Function) *For a square matrix $\boldsymbol{S}$, $\lambda(\boldsymbol{S}) = 1 - \min_{i_1,i_2}\left\{\sum_j \min\left(\boldsymbol{S}_{i_1 j}, \boldsymbol{S}_{i_2 j}\right)\right\}$.*

Having introduced the above, three Lemmas needed for the proofs provided in this paper are presented:

Lemma 1 *(Wolfowitz 1963) Let $\boldsymbol{M} = \{\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots\}$ be an infinite set of SIA matrices, where for each finite positive length sequence of $\boldsymbol{M}_{i_1}, \boldsymbol{M}_{i_2}, \ldots, \boldsymbol{M}_{i_j}$, the matrix product $\boldsymbol{M}_{i_j}\boldsymbol{M}_{i_{j-1}}\ldots\boldsymbol{M}_{i_1}$ is SIA. If for every $\boldsymbol{W} = \boldsymbol{M}_{k_1}\boldsymbol{M}_{k_2}\ldots\boldsymbol{M}_{N_t+1}$, where N_t is the number of different types of all SIA matrices of appropriate sizes, there exists a constant $0 \leq d < 1$ satisfying $\lambda(\boldsymbol{W}) \leq d$, then for each infinite sequence of $\boldsymbol{M}_{i_1}, \boldsymbol{M}_{i_2}, \ldots, \boldsymbol{M}_{i_j}, j \to \infty$ there exists a column vector $\boldsymbol{\nu}$ such that $\lim_{j\to\infty} \boldsymbol{M}_{i_j}\boldsymbol{M}_{i_{j-1}}\ldots\boldsymbol{M}_{i_1} = \mathbf{1}\boldsymbol{\nu}^{\mathsf{T}}$.*

Lemma 2 *(Ren, Beard, and McLain 2005) For $\mathcal{L}$ as a constant Laplacian matrix associated with a strongly connected network, the matrix $e^{-\mathcal{L}t}, \forall t > 0$ is a stochastic matrix with positive diagonal values that $\lim_{t\to\infty} e^{-\mathcal{L}t} = \mathbf{1}\nu^{\mathsf{T}}$ where $\boldsymbol{\nu} = [\nu_1, \nu_2, \ldots, \nu_n]^{\mathsf{T}} \geq 0$ and $\sum_{i=1}^{N} \nu_i = 1$.*

Lemma 3 *(Jadbabaie, Lin, and Morse 2003) Let $\boldsymbol{M}_1, \boldsymbol{M}_2, \ldots, \boldsymbol{M}_m$ be a finite set of non-negative matrices. Then $\boldsymbol{M}_1\boldsymbol{M}_2\ldots\boldsymbol{M}_m \geq \delta(\boldsymbol{M}_1 + \boldsymbol{M}_2 + \cdots + \boldsymbol{M}_m)$ for a δ which can be specified from matrices $\boldsymbol{M}_i, i \in \{1, 2, \ldots, m\}$.*

Methodology

Three sources of complexity arise in the mathematical analysis of CAIPD. Firstly, $\mathcal{L}(.)$ is time varying due to its nonlinear dependence on the state-variable $\boldsymbol{x}(t)$. Secondly, the system is highly state coupled, where many off-diagonal entries $\mathcal{L}_{ij}, i \neq j$ can take on arbitrary non-zero values. Finally, the analysis of Equation 1 resides in high dimensional spaces, rendering intuitive predictions difficult. These complexities are relaxed by considering the structure of $\mathcal{L}$ as a key for analysis, while relaxing its changes with respect to time. The main goal is then to determine for which network topologies convergence to an agreement, $\boldsymbol{x} \to \boldsymbol{x}^{\star}\mathbf{1}$ as $t \to \infty$ with $\mathbf{1} = [1, 1, \ldots, 1] \in \mathbb{R}^N$, occurs. To ensure the existence of an agreement, this paper uses the strong connectivity of social networks in which the graph $\mathbb{G}$ associated with $\mathbb{G}_{\boldsymbol{x}}$ has directed paths from any $v_i \in \mathcal{V}$ to $v_j \in \mathcal{V}$.

The analysis performed in this paper deals with two distinct scenarios with respect to the time varying nature of $\mathcal{L}$ in the dynamical model of Equation 1. Firstly, $\mathcal{L}$ matrix is assumed to be fixed which refers to the situations that individuals' don't update their beliefs based on each other's fitness. We refer to this as an *evolutionary network* where just the strategies evolve in time. Although such model pose a simplification of the original problem, its analysis can shed light on the dynamical behavior of the original system, where some of the attained results can be directly extended to the original problem. Secondly, the theoretical analysis is extended to the more general case of time varying Laplacian matrix, where both strategies and fitnesses evolve in time. We refer to this as a *coevolutionary network*. Empirical results confirming the provided theorems are also presented.

In what comes next, experiments are performed on two sample networks, shown in Figure 1, each consisting of $N = 50$ individuals. The scale-free network follows (Barabási and Albert 1999) and has an average degree of two; the small world network follows (Watts and Strogatz 1998) and has average degree 4 and rewiring probability 0.1. Both networks are initialized with 25 pure cooperators and 25 pure defectors. The benefits that cooperators share and the costs of cooperation are $b = 4$ and $c = 1$, respectively. The sigmoid function used to calculate the strategy adoption strength uses $\beta = 1$. Due to space constraints, additional empirical validations on other network sizes can be referred to in the supplementary material accompanying this paper.

Analysis of Evolutionary Networks

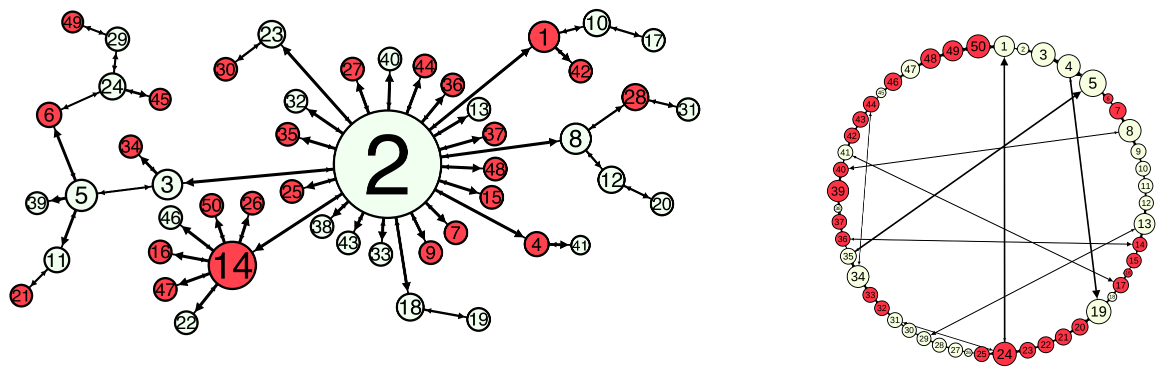
In evolutionary networks, the Laplacian matrix $\mathcal{L}$ is time-invariant. Therefore, CAIPD, in Equation (1), can be written in the general closed-loop control system form:

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \end{cases} \tag{2}$$

by setting $\boldsymbol{A} = \mathbf{0}$, $\boldsymbol{B} = -\boldsymbol{I}_N$, $\boldsymbol{C} = \boldsymbol{I}_N$ and $\boldsymbol{D} = \mathbf{0}$ with a feedback law of $\boldsymbol{u} = \mathcal{L}\boldsymbol{y}$.

Agreement in Evolutionary Networks

Convergence to the same value (i.e., $x_i(t) \to x_j(t),\ \forall (i,j) = 1, \ldots, N$ as $t \to \infty$) is first proved



(a) Scale free network

(b) Small world network

Figure 1: Sample networks; nodes with light and dark colors represent cooperators and defectors respectively. The size of the nodes represents their initial fitness value, while the size of the arrow heads are computed based on the corresponding elements of $\mathcal{L}$ matrix at the initial step, indicating the strength of influence between each set of nodes.

for evolutionary networks of form (2). An interpretation of the exact value of this agreement as a weighted average over all initial states is also derived. Further, it is shown that this value can be determined using the left eigenvector of the zero eigenvalue of the Laplacian matrix

Theorem 1 (Evolutionary Agreement) *For system* (2) *with* $\boldsymbol{A} = \boldsymbol{0}$*,* $\boldsymbol{B} = -\boldsymbol{I}_N$*,* $\boldsymbol{C} = \boldsymbol{I}_N$*,* $\boldsymbol{D} = \boldsymbol{0}$ *and* $\boldsymbol{u} = \mathcal{L}\boldsymbol{y}$*, every individual's state* x_i*,* $i = 1, 2, \ldots, N$ *converges to an agreement of the form* $x_i(t) \rightarrow \boldsymbol{r}^T\boldsymbol{x}(0)$*,* $t \rightarrow \infty$*, where* $\boldsymbol{r}$ *is the trivial left eigenvector of* $\mathcal{L}$ *(i.e., the eigenvector associated with the zero eigenvalue).*

Proof: As shown in (Olfati-Saber and Murray 2004, Theorems 1 and 2), a Laplacian matrix of a strongly connected digraph with N nodes, has $N-1$ eigenvalues with *positive real parts* and a *singular trivial eigenvalue* $\lambda_0 = 0$. The trivial right eigenvector of the Laplacian matrix is $\boldsymbol{1} = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$ and the trivial left eigenvector $\boldsymbol{r} = [r_1, r_2, \ldots, r_n]^\mathsf{T} \in \mathbb{R}^N$, where $\boldsymbol{r}^\mathsf{T}\boldsymbol{1} = 1$. First, $\mathcal{L}$ is mapped to the Jordan normal form as: $\mathcal{J} = \boldsymbol{T}_l\mathcal{L}\boldsymbol{T}_r$, with $\mathcal{J}$ being an upper triangular matrix having $\mathcal{J}_{11} = 0$, and $\mathcal{J}_{ii} = \lambda_j$ for $j = 1, 2, \ldots, N-1$ and $i = 2, 3, \ldots, N$. Further, $\boldsymbol{T}_l \in \mathbb{R}^{N\times N}$ contains the transpose of left eigenvectors of $\mathcal{L}$ with $\boldsymbol{r}^\mathsf{T}$ in the first row, and $\boldsymbol{T}_r \in \mathbb{R}^{N\times N}$ incorporates all right eigenvectors of $\mathcal{L}$ with $\boldsymbol{1}$ in the first column. Moreover, $\boldsymbol{T}_l\boldsymbol{T}_r = \boldsymbol{T}_r\boldsymbol{T}_l = \boldsymbol{I}_N$. Next, consider the state transformation $\tilde{\boldsymbol{x}} = \boldsymbol{T}_l\boldsymbol{x}$, with $\tilde{\boldsymbol{x}} = [\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N]^\mathsf{T}$. The system in Equation 2 can be represented in terms of $\tilde{\boldsymbol{x}}$ as: $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{T}_l\boldsymbol{B}\boldsymbol{u} = -\boldsymbol{T}_l\mathcal{L}\boldsymbol{T}_r\boldsymbol{T}_l\boldsymbol{x} = -\mathcal{J}\tilde{\boldsymbol{x}}$. The solution can be computed using $\tilde{\boldsymbol{x}}(t) = e^{-\mathcal{J}t}\tilde{\boldsymbol{x}}(0)$. It can be easily shown that $\tilde{\boldsymbol{x}}(t) \rightarrow [\tilde{x}_1(0), 0, 0 \ldots, 0]^\mathsf{T}$ as $t \rightarrow \infty$, with $\tilde{x}_1 = \boldsymbol{r}^\mathsf{T}\boldsymbol{x}(0)$. Using the state transformation $\boldsymbol{x} = \boldsymbol{T}_r\tilde{\boldsymbol{x}}$ it can be seen that $\boldsymbol{x}(t) \rightarrow \boldsymbol{T}_r\,[\tilde{x}_1(0)\;0\;0\ldots\;0]^\mathsf{T} = \boldsymbol{r}^\mathsf{T}\boldsymbol{x}(0)$, thus concluding the proof. □

Theorem 1 shows that all individuals in an evolutionary network eventually agree on the scalar state variable $\boldsymbol{r}^\mathsf{T}\boldsymbol{x}(0)$. Therefore, the final agreement is a weighted average of the initial states:

$$x_i(t) \rightarrow r_1x_1(0) + r_2x_2(0) + \ldots r_Nx_N(0) \tag{3}$$

for every i, with $\sum_{i=1}^{N} r_i = 1$.

Experimental Validation: According to the above theorem, pinpointing the individual(s) with the highest weights in (3) can help in approximating the final agreement of the network. For instance, considering the network in Figure 1(a), the elements of the trivial left eigenvector can be computed as $r_1 = 0.01, r_2 = 0.68, r_4 = 0.01, r_{14} = 0.30$, and $r_i \approx 0$ for every $i \neq 1, 2, 4, 14$. These are illustrated in Figure 2(a). Using the derived result of Equation (3), the final agreement is expected to be reached at $x^\star = 0.68$. Clearly, this is verified by Figure 2(b) presenting the evolution of state trajectories. Figures 2(c) and 2(d) again demonstrate these results in small world networks. The differences between the weight distributions are due to the fact that scale free networks possess a power-law degree distribution.

Based on the above results, next CAIPD is extended to allow for influencing the behavior of an evolutionary social network by incorporating modification to its action matrix model.

Agreement with Multi-rate State Updates

To allow weighted averaged manipulations, the action matrix $\boldsymbol{B}$ in Equation 2 is modified to $\boldsymbol{B}_m = \text{diag}(b_{m,i})\ \forall i \in \{1, 2, \ldots, N\}$, where $-1 \leq b_{m,i} < 0$, $\forall i \in 1, 2, \ldots, N$. $\boldsymbol{B}_m$ can be regarded as a multi-rate input matrix since as the absolute values of $b_{m,i}$ increase so does the variational speed of x_i, and vice versa. The following theorem, studies the agreement behavior of system (2) while using the multi-rate input matrix $\boldsymbol{B}_m$. It shows that all individuals eventually agree, even in the presence of such a multi-rate state update. It further demonstrates that individuals with lower update rates contribute more to the final agreement compared to those with a high update rate.

Theorem 2 (Evolutionary Multirate Agreement) *For system* (2) *with* $\boldsymbol{A} = \boldsymbol{0}$*, input matrix* $\boldsymbol{B}_m = diag(b_{m,i})$*,*

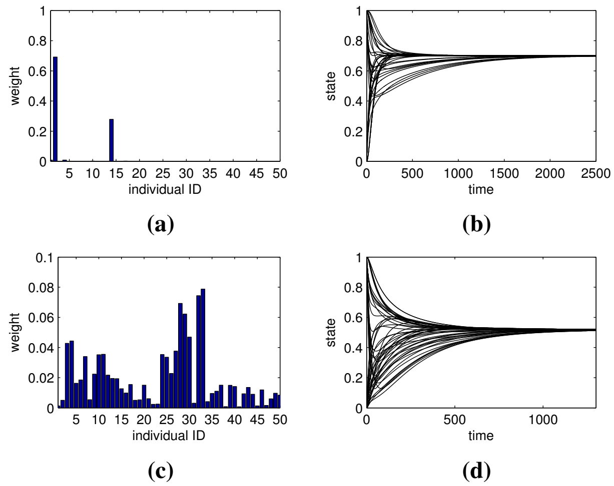


Figure 2: Agreement analysis for evolutionary networks **(a)-(b)** scale free network **(c)-(d)** small world network.

where $-1 \leq b_{m,i} < 0$, $\forall i \in 1, 2, \ldots, N$, $\boldsymbol{C} = \boldsymbol{I}_N$, $\boldsymbol{D} = \boldsymbol{0}$ *and input vector* $\boldsymbol{u} = \mathcal{L}\boldsymbol{y}$*, each individual's state* x_i*, for* $i = 1, 2, \ldots, N$ *converges to an agreement:*

$$x_i(t) \rightarrow \frac{1}{||\boldsymbol{r}^\top \boldsymbol{B}_m^{-1}||_1} \boldsymbol{r}^\top \boldsymbol{B}_m^{-1} \boldsymbol{x}(0), \text{ as } t \rightarrow \infty \quad (4)$$

where $\boldsymbol{r}$ *is the trivial left eigenvector of* $\mathcal{L}$ *and* $||\cdot||_1$ *denotes the* L_1 *norm.*

Proof: It can be easily shown that for any negative matrix $\boldsymbol{B}_m$, the $\mathcal{L}_m = \boldsymbol{B}_m \mathcal{L}$ still has one singular trivial eigenvalue associated, and its trivial right eigenvector will be $\boldsymbol{1}$. Furthermore, after some manipulations it can be seen that the normalized trivial left eigenvector of $\mathcal{L}_m$ is $\boldsymbol{r}_m = \frac{1}{||\boldsymbol{r}^\top \boldsymbol{B}_m^{-1}||_1} \boldsymbol{r}^\top \boldsymbol{B}_m^{-1}$. Please note that the non-trivial eigenvalues and eigenvectors might have changed from the original system. Following a similar procedure to that of Theorem 1, it becomes clear that as $t \rightarrow \infty$, the state of the i^{th} individual converges to: $x_i \rightarrow \frac{1}{||\boldsymbol{r}^\top \boldsymbol{B}_m^{-1}||_1} \boldsymbol{r}^\top \boldsymbol{B}_m^{-1} \boldsymbol{x}(0)$, thus concluding the proof. □

According to the above theorem, the final agreement of evolutionary networks with multi-rate action matrices can still be seen as a weighted average of the initial states: $x^\star = \left(\frac{r_1}{\alpha b_{m,1}} x_1(0) + \frac{r_2}{\alpha b_{m,2}} x_2(0) + \cdots + \frac{r_N}{\alpha b_{m,N}} x_N(0) \right)$, where $\alpha = ||\boldsymbol{r}^\top \boldsymbol{B}_m^{-1}||_1$.

This represents an important characteristic of the evolution of cooperation in social networks. Namely, the less the update rate $b_{m,i}$ of an arbitrary individual, the more its state will contribute to the final agreement value. Such a conclusion can be used to *control* the evolutionary behavior of these networks. For instance, if internally or externally one can decrease the update rate of an individual (or a group of individuals), consequently the state value of that individual (or group) will play a more prominent role in the overall group's agreement. Next, we make use of the above results to study the extreme case in which a network has to be driven to a stationary stable reference state. Firstly, the following action matrix: $\boldsymbol{B}_r = \text{diag}(b_{r,i})$ for $i = 1, 2, \ldots, N$ is introduced, with:

$$b_{r,i} = \begin{cases} -1 & \text{if } i \neq \text{ref.} \\ 0 & \text{if } i = \text{ref.} \end{cases}, \; i = 1, 2, \ldots, N \quad (5)$$

with "ref." being the number of the reference individual. The following theorem, shows that eventually all individual states will converge to $x_{\text{ref.}}$.

Theorem 3 (Evolutionary State-Reference Agreement) *For system* (2) *with* $\boldsymbol{A} = \boldsymbol{0}$*,* $\boldsymbol{B} = \boldsymbol{B}_r$ *as in* (5)*,* $\boldsymbol{C} = \boldsymbol{I}_N$*,* $\boldsymbol{D} = \boldsymbol{0}$ *and* $\boldsymbol{u} = \mathcal{L}\boldsymbol{y}$ *every individual's state* $x_i, i = 1, 2, \ldots, N$ *converges to an agreement* $x_i(t) \rightarrow x_{\textit{ref.}}$ *as* $t \rightarrow \infty$*.*

Proof: Here, the network is not strongly connected since the associated digraph to $\mathcal{L}_r = \boldsymbol{B}_r \mathcal{L}$ is not strongly connected. However, one directed spanning tree containing all the nodes of the graph with $v_{\text{ref.}}$ as its root, exists. In (Ren and Beard 2005, Corollary 1), it is shown that presence of such a spanning tree is enough for the Laplacian matrix to have a singular trivial eigenvalue and positive real parts for the other eigenvalues. Furthermore, it can be easily seen that the trivial right eigenvector of $\mathcal{L}_r$ is $\boldsymbol{1}$ and that the trivial left eigenvector of $\mathcal{L}_r$ is $\boldsymbol{r}_r = [r_{r,1}, r_{r,2}, \ldots, r_{r,N}]^\top$ with $r_{r,i} = 1$ if $i =$ ref. and $r_{r,i} = 0$ if $i \neq$ ref. Following a procedure similar to the proof of Theorem 1, it can be shown that as $t \rightarrow \infty$, the i^{th} individual state $x_i(t) \rightarrow x_{\text{ref.}}(t)$. □

Intuitively, the above results show that as an individual, i, insists on retaining its state and refuses to switch its initial value, eventually all others will arrive at an agreement with that i^{th} individual.

Experimental Validation: Theorems 2 and 3 are empirically demonstrated in two control experiments. In the first, the update rate of a cooperator is decreased. According to Theorem 2, as the cooperator's update rate decreases, its contribution to the final agreement increases. This fact is illustrated in Figures 3(a)-3(d). In the second experiment, Theorem 3 was tested, where one defector was chosen as the reference individual. Results illustrated in Figures 3(e) and 3(f) confirm the conclusions of Theorem 3 by showing that all individuals eventually converge to pure defection.

Agreement in Coevolutionary Networks

Here, coevolutionary networks with varying Laplacian are studied. Firstly, the concept of a dwell time τ (Jadbabaie, Lin, and Morse 2003) is used to re-write the dynamics of CAIPD as:

$$\dot{\boldsymbol{x}}(t) = -\mathcal{L}_k \boldsymbol{x}(t), \quad (6)$$

where $\mathcal{L}_k = \mathcal{L}[\boldsymbol{x}(k\tau)]$ for $k\tau < t < (k+1)\tau$ and $k = 1, 2, \ldots$. Clearly, as $\tau \rightarrow 0$, the system in Equation 6 collapses to (1). On the other hand, as $\tau \rightarrow \infty$, evolutionary networks, discussed in the previous section, can be derived as special cases. For any other τ, a coevolutionary network, studied here, is attained. Using the theory of matrix differential equations, the solution of (6) has the general form of:

$$\boldsymbol{x}(t) = \lim_{j \rightarrow \infty} e^{\mathcal{L}_j \tau} e^{\mathcal{L}_{j-1} \tau} \ldots e^{\mathcal{L}_0 \tau} \boldsymbol{x}(0) \quad (7)$$

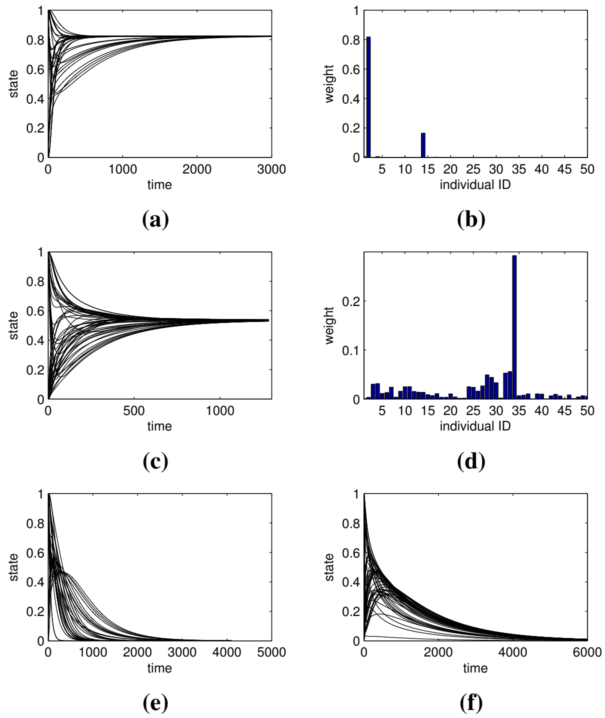


Figure 3: **(a)-(b)** scale-free network with multirate update $b_{m,2} = -0.5$ and $b_{m,i} = -1$ for every $i \neq 2$ **(c)-(d)** small world network with multirate update $b_{m,34} = -0.1$ and $b_{m,i} = -1$ for every $i \neq 34$ **(e)** scale-free network with state-reference agreement ref. $= 14$ **(f)** small world network with state-reference agreement ref. = 50.

where $\boldsymbol{x}(0)$ represents the initial network's configuration. Before studying the stability of (7), however, the following proposition reflects that strong node connectivity in $\mathcal{L}_k$, for any k, remains intact under $e^{-\mathcal{L}_k\tau}$.

Proposition 1 *If $\mathcal{L}_k$ is associated with a strongly connected network (i.e. there is a directional path between any two nodes), then $e^{-\mathcal{L}_k\tau}$ is strongly connected for every $\tau \in \mathbb{R}$.*

Proof: The matrix $\mathcal{L}_k$ is associated with a strongly connected network. Accordingly, it can be written as $\mathcal{L}_k = \boldsymbol{M} - d\boldsymbol{I}_n$, where $d = \max\{|\mathcal{L}_{k_{ii}}|\}$, with $\mathcal{L}_{k_{ii}}$ being the diagonal entries of $\mathcal{L}_k$. Therefore, $e^{-\mathcal{L}_k\tau} = e^{(\boldsymbol{M}-d\boldsymbol{I}_n)\tau} = e^{-d\boldsymbol{I}\tau}e^{\boldsymbol{M}\tau} \geq \delta\boldsymbol{M}$ for some $\delta > 0$. This shows that any two nodes with a direct link in $\boldsymbol{M}$ and $\mathcal{L}_k$ (i.e., $-\mathcal{L}_{k_{ij}} = \boldsymbol{M}_{ij} > 0$ and $-\mathcal{L}_{k_{ji}} = \boldsymbol{M}_{ji} > 0$) have a direct link in $e^{-\mathcal{L}_k\tau}$. Therefore, $e^{-\mathcal{L}_k\tau}$ is associated with a strongly connected network, thus concluding the proof. $\square$

Following the previous proposition and making use of Lemmas 2 and 3, a theorem showing that $\boldsymbol{x}(t)$ asymptotically converges to an agreement (i.e. $x_i(t) \to x_j(t), \forall i, j = 1, 2, \ldots, N$) is next presented and proven.

Theorem 4 (Coevolutionary Agreement) *For system* (2) *with $\boldsymbol{A} = \boldsymbol{0}$, $\boldsymbol{B} = -\boldsymbol{I}_N, \boldsymbol{C} = \boldsymbol{I}_N$, $\boldsymbol{D} = \boldsymbol{0}$ and $\boldsymbol{u} = \mathcal{L}_k\boldsymbol{y}$, where $k\tau < t < (k+1)\tau$ and τ is the dwell time, every individual's state $x_i, i = 1, 2, \ldots, N$ converges to an agreement as $x_i(t) \to x_j(t), t \to \infty$ for every i, j.*

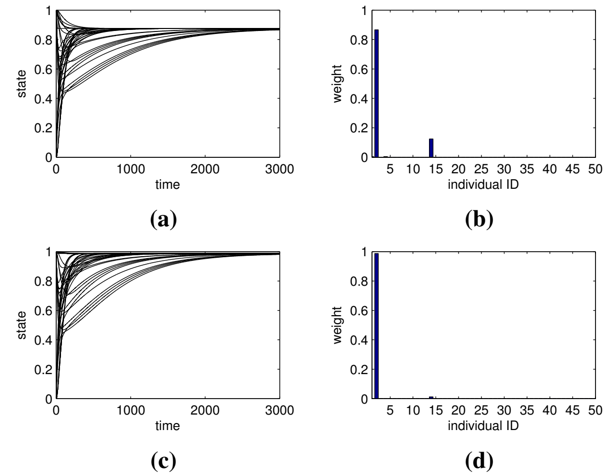


Figure 4: Agreement in coevolutionary scale free network with different dwell times **(a)-(b)** $\tau = 20$ **(c)-(d)** $\tau = 0.01$.

Proof: According to Lemma 2, $e^{\mathcal{L}_k\tau}$ converges to $\mathbf{1}\nu^\mathsf{T}$ for all k, with $\sum_{i=1}^{N}\nu_i = 1$. Further, it can be verified that $\lim_{n\to\infty}\left(e^{\mathcal{L}_k\tau}\right)^n = \mathbf{1}\nu^\mathsf{T}$. Therefore, the matrix $e^{\mathcal{L}_k\tau}$ is an SIA.

Using Lemma 3 and Proposition 1, it is clear that the state-transition matrix $\boldsymbol{\Psi} = e^{\mathcal{L}_m\tau}e^{\mathcal{L}_{m-1}\tau}\ldots e^{\mathcal{L}_0\tau}$ represents a strongly connected network. Furthermore, $\boldsymbol{\Psi}$ is stochastic, therefore, according to Lemma 2, such a matrix is SIA.

Note that according to (Ren, Beard, and Kingston 2005, Proof of Theorem 3.2) and the fact that the Laplacian matrices $\mathcal{L}_k$ for every k share the same spanning trees through the coevolution, the condition required for convergence of a sequence of an infinite number of SIA matrices in Lemma 1 (i.e., $\lambda(.) \leq d, 0 \leq d < 1$) holds. Therefore, it can be proven that $\lim_{j\to\infty} e^{\mathcal{L}_j\tau}e^{\mathcal{L}_{j-1}\tau}\ldots e^{\mathcal{L}_0\tau} = \mathbf{1}\nu^\mathsf{T}$, with ν being a column vector summing to one. Hence

$$\boldsymbol{x}(t) = \lim_{j\to\infty} e^{\mathcal{L}_j\tau}e^{\mathcal{L}_{j-1}\tau}\ldots e^{\mathcal{L}_0\tau}\boldsymbol{x}(0) = \mathbf{1}\nu^\mathsf{T}\boldsymbol{x}(0) = x^\star\mathbf{1}, \tag{8}$$

where $x^\star \in \mathbb{R}$ denotes the agreement point, thus concluding the proof. $\square$

Theorem 4 shows that the time varying Laplacian matrix, represented by the dynamical system of Equation (1) converges to an agreement: $x_i(t) \to x^\star$ as $t \to \infty$ for every i, with $x^\star$ being a weighted average of $x_i, i = 1, 2, \ldots, N$.

Experimental Validation: Figures 4 and 5 illustrate the evolution of the sample networks of Figure 1 for two different dwell times. Furthermore, the elements of ν in Equation 8 are also shown. Clearly, Theorem 4 is validated since an agreement can be asymptotically reached. Moreover, it can be seen that systems of exactly the same initial configurations but different dwell times can have unequal final agreements.

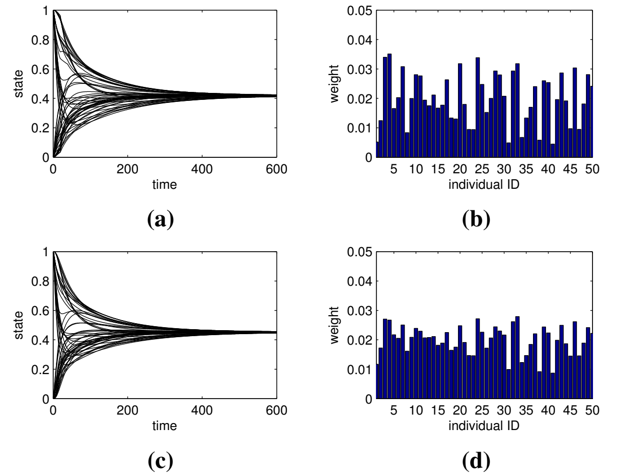


Figure 5: Agreement in coevolutionary small world network with different dwell times **(a)-(b)** $\tau = 20$ **(c)-(d)** $\tau = 0.01$.

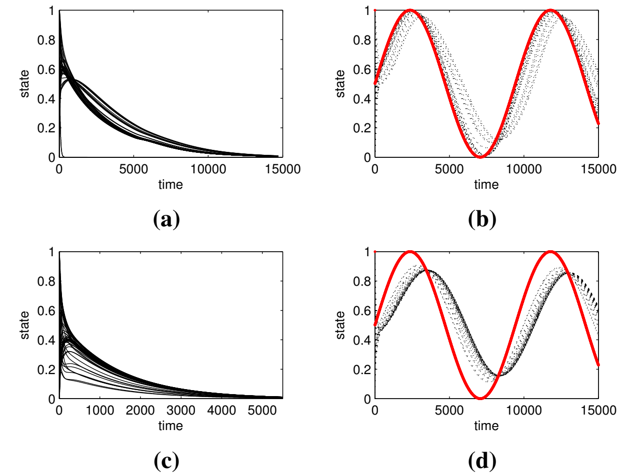


Figure 6: State-reference agreement in coevolutionary networks **(a)** scale-free network with ref.= 14 **(b)** scale-free network with ref.=2 and $x_2(t)$ smoothly changing between cooperation and defection **(c)** small world network with ref.= 34 **(d)** small world network with ref.=50 and $x_{50}(t)$ smoothly changing between cooperation and defection.

Although no closed form solution can be derived for the multi-rate agreement of *coevolutionary* networks, next a theorem for state-reference agreement showing that an individual with a fixed state inevitably determines the final agreement value is presented and proved.

Theorem 5 (Coevolutionary State-Reference Agreement) *For system* (2) *with* $\boldsymbol{A} = \boldsymbol{0}$*,* $\boldsymbol{B} = \boldsymbol{B}_r$ *as in* (5)*,* $\boldsymbol{C} = \boldsymbol{I}_N$*,* $\boldsymbol{D} = \boldsymbol{0}$ *and* $\boldsymbol{u} = \mathcal{L}_k \boldsymbol{y}$*, where* $k\tau < t < (k+1)\tau$ *and* τ *being the dwell time, every individual's state* $x_i, i = 1, 2, \ldots, N$ *converges to an agreement as* $x_i(t) \to x_{\text{ref.}}, t \to \infty$*.*

Proof: Equation (2) is rewritten as:

$$\dot{\boldsymbol{x}} = \boldsymbol{B}_r \mathcal{L}_k \boldsymbol{x} \tag{9}$$

The solution of (9) can be expressed as

$$\boldsymbol{x}(t) = \lim_{j \to \infty} e^{\boldsymbol{B}_r \mathcal{L}_j \tau} e^{\boldsymbol{B}_r \mathcal{L}_{j-1} \tau} \ldots e^{\boldsymbol{B}_r \mathcal{L}_0 \tau} \boldsymbol{x}(0)$$

According to the structure of $\boldsymbol{B}_r$, it can be easily checked that $x_{\text{ref.}}(t) = x_{\text{ref.}}(0), \forall t > 0$ (the power series can be used to see that in the overall state-transition matrix of (2) all elements in the row corresponding to the reference individual are zero except the diagonal value which is one). Furthermore, as described in the proof of Theorem 5 the network associated with system (9) contains a spanning tree through the coevolution, and each $\mathcal{L}_k$ matrix has one singular trivial eigenvalue and positive real parts for the nontrivial eigenvalues. Therefore, following a similar procedure to that proving Theorem 4, it can be shown that all state variables coverage to a final agreement as: $x_{\text{ref.}}(t) = x_{\text{ref.}}(0)$ for $\forall t > 0$. □

Experimental Validation: In Figures 6(a) and 6(c) one defector is chosen as the reference, and it can be seen that all the individuals of the coevolutionary network eventually agree on pure defection. This verifies the results of Theorem 5.

To reflect upon the potential extension of the introduced framework, a tracking scenario is designed. A cooperator i is chosen as the reference state. However, its strategy varies, say according to $x_i(t) = \frac{1}{2} + \frac{1}{2}\sin(\frac{t}{1500})$. It is clear from Figures 6(b) and 6(d) that the whole network follows the reference state throughout the coevolution. The phase-shift observed for the small world network can be explained by the absence of hubs, causing the behavior to spread slowly.

Conclusion

This work has thoroughly analyzed and extended the CAIPD model, thereby gaining a broader understanding of the evolution of cooperation on complex social networks. Distinguishing between *evolutionary* networks, in which the interaction dynamics are fixed, and the more general case of *coevolutionary* networks with time-varying dynamics, three main contributions can be listed. Firstly, convergence to agreement in evolutionary networks has been proven (Theorem 1). Moreover, it has been proven that this final agreement is a weighted average of the initial state, and that these weights can be computed explicitly using the trivial left eigenvector of the Laplacian matrix associated with the network in the very first iteration. Secondly, these proofs have been extended to the more general case of coevolutionary networks (Theorem 4). Thirdly, an extension to CAIPD has been proposed that allows to model influence of the evolution of social networks towards states of specific individuals. It has been proven that individuals with lower adaptation rates contribute most to the final agreement. Moreover, all proofs have been validated empirically for both scale-free and small world networks.

These results provide a first step towards active control of complex social networks, by studying how certain individuals may influence the convergence and final agreement reached in the network.

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