2015 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining

On the Skewed Degree Distribution of Hierarchical Networks

Bijan Ranjbar-Sahraei
Maastricht University, The Netherlands
Email: b.ranjbar@ieee.org

Haitham Bou Ammar
University of Pennsylvania, United States
Email: haithamb@seas.upenn.edu

Karl Tuyls
University of Liverpool, United Kingdom
Email: k.tuyls@liverpool.ac.uk

Gerhard Weiss
Maastricht University, The Netherlands
Email: gerhard.weiss@maastrichtuniversity.nl

***Abstract*—In this paper, a prestige-based evolution process is introduced, which provides a formal framework for the study of linear hierarchies seen in human societies. Due to the deterministic characteristics of the proposed model, we are capable of determining equilibria in closed form. Surprisingly, these stationary points recover the power-law degree distribution as the shared property of the resulting hierarchal networks, explaining the prevalence of hierarchies in societies. This result sheds light on the evolutionary advantages of hierarchies.**

I. Introduction

To analyze the emergence of social networks, a variety of mathematical models have been proposed. The earliest dates back to the 1900's, where Yule [1] studied the biological evolution of species based on age and population data. Others, e.g., Lotka [2] provided rules required for describing and analyzing scientific publications. Resulting from these and other studies, was the emergence of the power-law degree distribution [3] as a shared common characteristic for a wide-range of networks including but not limited to, the world wide web, protein-protein interaction, airlines and social networks.

Given such a widely-shared characteristic, Barabási and Albert suggested a preferential attachment model for the generation of scale-free graphs exhibiting a power-law degree distribution [4]. As noted by Durret [5], the definition of their process was rather informal. Since then, different precise forms of the Barabási-Albert model have been studied in literature [6]. Though successful at recovering the power-law degree distribution, these studies impose several restricting assumptions on the underlying graph generating process. For instance, such techniques typically adopt a binary attachment model, in which two nodes are either connected or not [4], [7].

On the other hand, the existence of hierarchical relationships is another shared common characteristic for a wide-range of networks. Research has shown that human physique (e.g., stature) and body hormones play a crucial role in enabling dominance in the society. Contrary to animal societies which base hierarchies on dominance, however, human societies replace dominance by "prestige" to construct reciprocal relationships between leaders and followers [8]. Thus, evolutionary considerations of real-world networks suggests the emergence of scale-free behavior (i.e., networks exhibiting a power-law degree distribution) in networks as a resultant of hierarchal attachment processes which are not reflected through current preferential attachment models. Apart from this modeling restriction, another problem inherent to existing binary models lies in their explanatory capabilities. For instance, they fail to manifest connection strengths between individuals; a property being at the core of behavioral emergence in real networks [9].

To provide more realistic modeling outcomes, in this paper, we contribute by proposing a *deterministic* hierarchal graph attachment process for prestige-based human societies. Contrary to preferential attachment models, our approach only assumes hierarchal connections between individuals, thus bridging the modeling gap to real-world evolutionary networks. Among many advantages, our deterministic setting enables the derivation of the degree distribution in closed-form. Performing this derivation recovers, surprisingly, the power-law degree distribution as the main property of the resultant hierarchal networks, which explains the prevalence of such hierarchies in societies.

II. Notations

A network is described via a graph, $\mathbb{G} = (\mathbb{V}, \mathbb{W})$, consisting of a set of N nodes (or vertices) $\mathbb{V} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_N\}$ and an $N \times N$ adjacency matrix $\mathbb{W} = [\boldsymbol{w}_{ij}]$ where non-zero entries $\boldsymbol{w}_{ij}$ indicate the weighted connection from $\boldsymbol{v}_j$ to $\boldsymbol{v}_i$. In this paper we consider the undirected graphs which have symmetrical $\mathbb{W}$. The neighborhood $\mathbb{N}$ of a node $\boldsymbol{v}_i$ is defined as the set formed by its connected vertices, i.e., $\mathbb{N}(\boldsymbol{v}_i) = \cup_j \boldsymbol{v}_j : \boldsymbol{w}_{ij} > 0$.

The node's degree, $\deg(\boldsymbol{v}_i)$, is given by the cardinality of its neighborhood. The strength of a node is of major importance in hierarchical networks. Next, we define three concepts: (1) relative strength (2) strength observation and (3) absolute strength.

Relative Node Strength: The relative strength of j^{th} node with respect to i^{th} node with $i > j$ is denoted by $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$ and represents the sum over all edge weights between j^{th} node and every k^{th} node where $k < i$. Namely,

$$\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \sum_{k=1}^{i-1} \boldsymbol{w}_{jk}, \quad i > j. \tag{1}$$

ASONAM '15, August 25-28, 2015, Paris, France
 ISBN 978-1-4503-3854-7/15/08 $15.00
DOI: http://dx.doi.org/10.1145/2808797.2809409

In other words, when node i is monitoring node j with $j < i$, it just observes those connections from other ks to j which $k < i$.

Strength Observation The Strength observation of the i^{th} node is denoted by the vector

$$\vec{\mathbf{\Psi}}_i = [\mathbf{\Psi}_i(\boldsymbol{v}_1), \mathbf{\Psi}_i(\boldsymbol{v}_2), \ldots, \mathbf{\Psi}_i(\boldsymbol{v}_{i-1})]^{\mathsf{T}}$$

with cardinality $i - 1$.

Absolute Node Strength: The absolute strength of i^{th} node is defined as

$$\mathbf{\Psi}(\boldsymbol{v}_i) = \sum_{k=1}^{N} \boldsymbol{w}_{ik}. \tag{2}$$

III. Network Dynamics

Here, we propose a dynamical process which captures the edge dynamics of a complete network. Let $\boldsymbol{\omega} = \{w_{ij} | \forall i, j = 1, 2, ..., N, i > j\}$ denote the state vector of the process, where each state variable w_{ij} corresponds to the weight of the link between the j^{th} and i^{th} node. In the very general case, one considers the rate of changes in w_{ij} as a function of all state variables

$$\dot{w}_{ij} = f(\boldsymbol{\omega}) \tag{3}$$

In this paper, however, the focus is on hierarchical networks in which for any $i > j$, $\dot{w}_{ij}$ is a function of w_{ij} itself and the strength observation of the i^{th} node $\vec{\mathbf{\Psi}}_i$

$$\dot{w}_{ij} = f_{\Psi}(w_{ij}, \vec{\mathbf{\Psi}}_i), i > j \tag{4}$$

In other words, the dynamics of the linking strength between i and j is independent of any other node l where l is higher than i or j in the hierarchy.

Using f_{Ψ} from Equation 4, and sorting the state variables w_{ij} increasingly (based on $Ni+j$), the overall dynamic process can be written as

$$\begin{aligned}\dot{\boldsymbol{\omega}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[w_{21}, \ldots, w_{N(N-1)}\right]^{\mathsf{T}} \\ &= \left[f_{\Psi}(w_{21}, \vec{\mathbf{\Psi}}_2), \ldots, f_{\Psi}(w_{N(N-1)}, \vec{\mathbf{\Psi}}_N)\right]^{\mathsf{T}}\end{aligned}$$

Next, we introduce a possible strategy for $f_{\Psi}(.)$, namely $f_{\Psi}^{(\mathbb{P})}(\cdot)$ which represents the Prestige-based attachment (PA) models.

IV. Prestige-based Attachment Model

The overall strength of node i in establishing connection with every other j^{th} node is assumed to be limited and sums-up to 1. The prestige-based attachment model can be formally derived as follows. Let

$$\dot{w}_{ij} = f_{\Psi}^{(\mathbb{P})}(w_{ij}, \mathbf{\Psi}_i(\boldsymbol{v}_j), ||\vec{\mathbf{\Psi}}_i||)$$

and

$$f_{\Psi}^{(\mathbb{P})}(w_{ij}, \mathbf{\Psi}_i(\boldsymbol{v}_j), ||\vec{\mathbf{\Psi}}_i||) = \frac{\mathbf{\Psi}_i(\boldsymbol{v}_j)}{||\vec{\mathbf{\Psi}}_i||} - w_{ij}, \quad j < i \tag{5}$$

where $||.||$ denotes the first norm.

By studying the dynamic process proposed in Equation 5, it can be easily seen that $\dot{w}_{ij}, i > j$ is a function of every w_{kl} for $k, l < i$. Without loss of generality we assume $\boldsymbol{w}_{11}^{(\mathbb{P})} = 1$, such that

$$\mathbf{\Psi}_2(\boldsymbol{v}_1) = 1 \tag{6}$$

and $\boldsymbol{w}_{ii}^{(\mathbb{P})} = 0$ for every $i > 1$. It is straightforward to compute the equilibrium point of this system

$$\boldsymbol{w}_{ij}^{(\mathbb{P})} = \frac{\mathbf{\Psi}_i(\boldsymbol{v}_j)}{||\vec{\mathbf{\Psi}}_i||} \tag{7}$$

The equilibrium point (7) explains that the connection strength between node i and node j depends on the connection strength among nodes i and j and every k^{th} node with $k < max(i, j)$. To illustrate, imagine N agents who are all connected to each other, and continuously the agents with higher order share their available resources with agents with lower order. The strength of link between i and j shows the amount of resources which are transmitted from i to j. According to (7) the 2^{nd} individual shares all of her resources with 1^{st} individual (i.e., $w_{21} = \frac{1}{1} = 1$). The 3^{rd} individual shares one third of her resources with the 2^{nd} individual and two third of it with the 1^{st} individual (i.e., $w_{32} = \frac{1}{1+2} = \frac{1}{3}$ and $w_{31} = \frac{2}{1+2} = \frac{2}{3}$). With the same respect, i^{th} individual shares portions of her resources with each of the j individuals where $j < i$, while those with lower order receive more. We call this a *prestige-based model* as the lower orders reflect a kind of prestige in the group and high prestige agents receive more than agents with lower prestige.

An immediate result of (7) is that

$$\sum_{j=1}^{i} \boldsymbol{w}_{ij}^{(\mathbb{P})} = \sum_{j=1}^{i-1} \boldsymbol{w}_{ij}^{(\mathbb{P})} = \sum_{j=1}^{i-1} \frac{\mathbf{\Psi}_i(\boldsymbol{v}_j)}{||\vec{\mathbf{\Psi}}_i||} = \frac{||\vec{\mathbf{\Psi}}_i||}{||\vec{\mathbf{\Psi}}_i||} = 1. \tag{8}$$

Next, we study the amount of resources each individual receives in such prestige-based network (captured by node's strengths), and also compute the distribution of node strengths.

A. Analysis of Node's Strength

Here we determine a closed form solution for the sum over the strength of every j^{th} node from the perspective of the i^{th} node, where $j < i$.

Lemma 1: In the prestige-based attachment model, the sum of the relative node strengths of every j^{th} node from perspective of the i^{th} node, where $j < i$ is as following

$$\boldsymbol{K}(i) : \|\vec{\mathbf{\Psi}}_i\| = 2i - 3.$$

Proof: The above lemma can be proved by using induction:

Initial Step: According to Equation (6) we have $\|\vec{\mathbf{\Psi}}_2\| = \mathbf{\Psi}_2(\boldsymbol{v}_1) = 1$. Therefore, $\boldsymbol{K}(i)$ holds for $i = 2$.

Inductive Step: Let

$$\boldsymbol{K}(i-1) : \|\vec{\mathbf{\Psi}}_{i-1}\| = 2i - 5,$$

and also note that $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. Therefore we can write

$$\begin{aligned}\|\vec{\boldsymbol{\Psi}}_i\| =& \sum_{j=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_j)\\ =& \boldsymbol{\Psi}_i(\boldsymbol{v}_{i-1}) + \sum_{j=1}^{i-2}\left(\boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}\right)\\ =& \sum_{j=1}^{i-1} \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} + \|\vec{\boldsymbol{\Psi}}_{i-1}\| + \sum_{j=1}^{i-2} \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}\end{aligned}$$

By using $\boldsymbol{K}(i-1)$ and Equation 8, we'll get

$$\|\vec{\boldsymbol{\Psi}}_i\| = \sum_{j=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 1 + 2i - 5 + 1 = 2i - 3 \tag{9}$$

Therefore, $\boldsymbol{K}(i)$ holds for every i, concluding the proof. ∎

B. Analysis of Edge Weights

We can compute the edge weight between i^{th} node and j^{th} node as follows.

Lemma 2 (Edge Weight): For the weighted graph $\mathbb{G}$, evolved with PA model, i^{th} node is connected to j^{th} node with an edge of weight

$$\boldsymbol{K}(i) : \boldsymbol{w}^{(\mathbb{P})}_{ij} = \frac{1}{2i-2} \prod_{k=1}^{i-j} \frac{2i-2k}{2i-2k-1}, \forall j < i. \tag{10}$$

Proof:

The validity of Equation 10 can be proved for each i and for every $j < i$ using induction.

Initial Step: The second node is connected to the first node with $\boldsymbol{w}^{(\mathbb{P})}_{21} = 1$, meaning that $\boldsymbol{K}(2)$ holds.

Inductive Step: Now assume that

$$\boldsymbol{K}(i-1) : \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} = \frac{1}{2i-4} \prod_{k=1}^{i-j-1} \frac{2i-2k-2}{2i-2k-3}$$

holds for every $j < i-1$. For computing the edge weight between i^{th} and j^{th} node, recall that $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. By using (7) and Lemma 1, it can be seen that:

$$\begin{aligned}\boldsymbol{\Psi}_i(\boldsymbol{v}_j) &= \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}\\ &= (2i-5)w_{(i-1)j} + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}\\ &= (2i-4)w_{(i-1)j}\end{aligned} \tag{11}$$

Using Equations 7, 11 and Lemma 1, the edge weight between i^{th} and j^{th} nodes can be written as

$$\boldsymbol{w}^{(\mathbb{P})}_{ij} = \frac{\boldsymbol{\Psi}_i(\boldsymbol{v}_j)}{\sum_{k=1}^{i-1} \boldsymbol{\Psi}_i(\boldsymbol{v}_k)} = \frac{1}{2i-2} \prod_{k=1}^{i-j} \frac{2i-2k}{2i-2k-1}$$

for $j < i-1$. Therefore, $\boldsymbol{K}(i)$ holds for every i, concluding the proof. ∎

Before, computing the distribution of strengths for the PA model, we present the following proposition providing the relative strength of the j^{th} node from the perspective of the i^{th} for every $i > j$ (i.e., $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$) in closed form.

Proposition 1 (Relative Node Strength): For the weighted graph $\mathbb{G}$, evolved according to the PA model, the strength of the j^{th} node from perspective of the i^{th} node is given by

$$\boldsymbol{K}(i) : \begin{cases} \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \prod_{k=j+2}^{i} \frac{2k-4}{2k-5} & \text{for } j < i-1 \\ \boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 1 & \text{for } j = i-1. \end{cases} \tag{12}$$

Proof: Again, induction can be used to prove the validity of Equation 12. Starting with the initial step we get

Initial Step: From Equation 6, the strength of the first node from the perspective of the second node is $\boldsymbol{\Psi}_2(\boldsymbol{v}_1) = 1$. Besides, using Lemma 2 we can deduce that

$$\boldsymbol{\Psi}_3(\boldsymbol{v}_1) = \frac{\boldsymbol{w}^{(\mathbb{P})}_{11} + \boldsymbol{w}^{(\mathbb{P})}_{21}}{3} = \frac{2}{3}.$$

Therefore, $\boldsymbol{K}(2)$ holds. For the inductive step we proceed as follows

Inductive Step: Assume that following holds.

$$\boldsymbol{K}(i-1) : \begin{cases} \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) = \prod_{k=j+2}^{i-1} \frac{2k-4}{2k-5} & \text{for } j < i-2 \\ \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) = 1 & \text{for } j = i-2. \end{cases}$$

For computing $\boldsymbol{\Psi}_i(\boldsymbol{v}_j)$, consider $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j}$. Using Equation 7 and Lemma 1, we can show that for every $j < i-1$

$$\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = \boldsymbol{\Psi}_{i-1}(\boldsymbol{v}_j) + \boldsymbol{w}^{(\mathbb{P})}_{(i-1)j} = \prod_{k=j+2}^{i} \frac{2k-4}{2k-5}$$

Besides using Equation (8), $\boldsymbol{\Psi}_i(\boldsymbol{v}_j) = 1$ for $j = i-1$. Therefore, $\boldsymbol{K}(i)$ holds for every i and the proof is concluded. ∎

Lemma 3 (Global Strength): For the weighted graph $\mathbb{G}$, evolved with the PA model, the global strength of the i^{th} node is

$$\begin{cases} \boldsymbol{\Psi}(\boldsymbol{v}_i) = \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5} & \text{for } i < N \\ \boldsymbol{\Psi}(\boldsymbol{v}_i) = 1 & \text{for } i = N. \end{cases} \tag{13}$$

Proof: We know that $\boldsymbol{\Psi}(\boldsymbol{v}_i) = \boldsymbol{\Psi}_N(\boldsymbol{v}_i) + \boldsymbol{w}^{(\mathbb{P})}_{iN}$ for every $i < N$. Using Equation 7 and Proposition 1, we have

$$\begin{aligned}\boldsymbol{\Psi}(\boldsymbol{v}_i) &= \boldsymbol{\Psi}_N(\boldsymbol{v}_i) + \frac{\boldsymbol{\Psi}_N(\boldsymbol{v}_i)}{2N-3}\\ &= \frac{2N-2}{2N-3} \prod_{k=i+2}^{N} \frac{2k-4}{2k-5}\\ &= \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5}\end{aligned}$$

for every $i < N$. Based on Equation (8) we have

$$\boldsymbol{\Psi}(\boldsymbol{v}_N) = \sum_{i=1}^{N-1} \boldsymbol{w}^{(\mathbb{P})}_{Ni} = 1.$$

This concludes the proof. ∎

Finally, we can compute the strength distribution in a closed form. In analysis of weighted networks, typically the Distribution Function (DF) of node strengths is defined as

$$P(k) = \left\| \left\{ v_i | \forall i, k \leq \boldsymbol{\Psi}(\boldsymbol{v}_i) < k+1 \right\} \right\| \tag{14}$$

where $\boldsymbol{\Psi}(\boldsymbol{v}_i)$ defined in (2) denotes the strength of node v_i.

To ease the analysis, in this work we make use of the Complementary Cumulative Distribution Function (CCDF) defined as:

$$P_c(k) = \left| \left\{ v_i | \forall i, \boldsymbol{\Psi}(\boldsymbol{v}_i) \geq k \right\} \right| \tag{15}$$

The following lemma clarifies the relation between the DF and CCDF for a network with power-law Distribution.

Lemma 4 (Power-law Distribution): Consider a power-law distribution in form of $P(k) = ck^{-\alpha}$, where α is the power-law exponent. The CCDF $P_c(k)$ also follows a power-law but with an exponent $\alpha - 1$.

Proof: Can be easily seen by simple integration. ■

The following theorem provides the strength distribution of a PA model.

Theorem 1 (Strength Distribution): For the weighted graph $\mathbb{G}$ evolved with the PA model, the distribution of the global strength k follows a power-law with exponent -3

$$P(k) \propto k^{-3}.$$

For proving Theorem 1, we need the following lemma, first.

Lemma 5 (Fraction Product Series): Consider the following product of fractions

$$L(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-4}{2k-5},$$

then $\gamma i^{-\frac{1}{2}} < L(i,N) < \gamma(i-1)^{-\frac{1}{2}}$, with $\gamma = \sqrt{N-1}$

Proof: We use the comparison test to compute the lower and upper bounds of $L(i,N)$. Firstly, consider

$$Q(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-5}{2k-6} \tag{16}$$

Clearly, $L(i,N) < Q(i,N)$ and $L(i,N)Q(i,N) = \frac{2N-2}{2i-2}$. Therefore, $L(i,N) < \sqrt{\frac{2N-2}{2i-2}} \leq \gamma(i-1)^{-\frac{1}{2}}$ concluding the upper-bound. To determine the lower bound, define

$$Q'(i,N) = \prod_{k=i+2}^{N+1} \frac{2k-3}{2k-4} \tag{17}$$

It can be shown that $L(i,N) > Q'(i,N)$ and $L(i,N)Q'(i,N) = \frac{2N-1}{2i-1}$. Therefore,

$$L(i,N) > \sqrt{\frac{2N+1}{2i-1}} > \sqrt{\frac{2N-2}{2i}} \geq \gamma i^{-\frac{1}{2}}. \tag{18}$$

■

Using results of Lemmas 3, 4 and 5 we give the following proof for Theorem 1.

Proof: The following lower and upper bounds can be computed for the strength of the i^{th} node

$$\gamma i^{-\frac{1}{2}} < \boldsymbol{\Psi}(\boldsymbol{v}_i) < \gamma(i-1)^{-\frac{1}{2}} \tag{19}$$

where $\gamma = \sqrt{N-1}$. From Equation (19), we have

$$\boldsymbol{\Psi}(\boldsymbol{v}_i) \geq k, \text{ for } i \in \{1,2,3,\ldots,\left\lfloor \frac{\gamma^2}{k^2} \right\rfloor\} \tag{20}$$

$$P_c(k) = \left| \left\{ 1,2,3,\ldots,\frac{\gamma^2}{k^2} \right\} \right| \simeq \gamma^2 k^{-2} \tag{21}$$

Therefore,

$$P_c(k) \propto k^{-2} \tag{22}$$

Using Lemma 4, we have

$$P(k) \propto k^{-3}, \tag{23}$$

thus proof is concluded. ■

V. Conclusion

In this paper we proposed a dynamical model for prestige-based hierarchical systems. Although, the dynamical system was described using simple hierarchical rules, the derived stationary points had been shown to recover a *power-law* degree distribution with exponent -3 (Theorem 1). Emergence of this degree distribution despite the simple hierarchical structure explains how hierarchical social structures have survived in human societies. There are various interesting future directions of this work. We plan to validate the attained results through data gathered from real-world networks. Moreover, our model is proposed in form of a dynamical process, which makes it possible for the development of control strategies. The overall idea would be the control of the evolution of the network to arrive at specific network forms.

References

[1] G. U. Yule, "A mathematical theory of evolution, based on the conclusions of dr. jc willis, frs," *Philosophical Transactions of the Royal Society of London. Series B, Containing Papers of a Biological Character*, pp. 21–87, 1925.

[2] A. J. Lotka, "The frequency distribution of scientific productivity." *Journal of Washington Academy Sciences*, 1926.

[3] R. F. i. Cancho and A. H. Fernández, "Power laws and the golden number," *Problems of general, germanic and slavic linguistics*, pp. 518–523, 2008.

[4] A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–12, Oct. 1999. [Online]. Available: http://www.ncbi.nlm.nih.gov/pubmed/10521342

[5] R. Durrett, *Random Graph Dynamics (Cambridge Series in Statistical and Probabilistic Mathematics)*. New York, NY, USA: Cambridge University Press, 2006.

[6] B. Bollobás and O. Riordan, "Coupling scale-free and classical random graphs," *Internet Mathematics*, vol. 1, no. 2, pp. 215–225, 2003. [Online]. Available: http://projecteuclid.org/euclid.im/1089229509

[7] D. J. Watts and S. H. Strogatz, "Collective dynamics of 'small-world' networks," *nature*, vol. 393, no. 6684, pp. 440–442, 1998.

[8] M. E. Price and M. Van Vugt, "The evolution of leader–follower reciprocity: the theory of service-for-prestige," *Frontiers in human neuroscience*, vol. 8, 2014.

[9] A. Barrat, M. Barthélemy, and A. Vespignani, "Modeling the evolution of weighted networks," *Physical Review E*, vol. 70, no. 6, p. 066149, 2004.